

NON-INNER AUTOMORPHISMS OF ORDER p IN FINITE NORMALLY CONSTRAINED p -GROUPS

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ABSTRACT. In this paper we study the existence of at least one non-inner automorphism of order p in a finite normally constrained p -group when p is an odd prime.

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1. INTRODUCTION

The main goal of this paper is to contribute to the longstanding conjecture of Berkovich posed in 1973, that conjectures that every finite p -group admits a non-inner automorphism of order p , where p denotes a prime number [19, Problem 4.13]. The conjecture has attracted the attention of many mathematicians during the last couple of decades, and has been confirmed for many classes of finite p -groups. It is remarkable to put on record that, in 1965, Liebeck [15] proved the existence of a non-inner automorphism of order p in all finite p -groups of class 2, where p is an odd prime. However, the fact that there always exists a non-inner automorphism of order 2 in all finite 2-groups of class 2 was proved by Abdollahi [1] in 2007. The conjecture was confirmed for finite regular p -groups by Schmid [18] in 1980. Indeed, Deaconescu [12] proved it for all finite p -groups G which are not strongly Frattinian. Moreover, Abdollahi [2] proved it for finite p -groups G

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such that $G/Z(G)$ is a powerful p -group, and Jamali and Visesh [14] did the same for finite p -groups with cyclic commutator subgroup. In the realm of finite groups, quite recently, the result has been confirmed for semi-abelian p -groups by Benmoussa and Guerboussa [6], and for p -groups of nilpotency class 3, by Abdollahi, Ghorraishi and Wilkens [3]. To be more precise, Abdollahi and Ghorraishi in [4] proved that in some cases the non-inner automorphism of order p can be chosen so that it leaves $Z(G)$ elementwise fixed. Finally, Abdollahi et al [5] proved the conjecture for p -groups of coclass 2, and quite recently in [17] M.Ruscitti, L. Legarreta and M.K.Yadav did the same for p -groups of coclass 3 when p is a prime different from 3.

With the contribution of this paper we add basically another class of finite p -groups to the above list, by proving that the above mentioned conjecture holds true for all finite normally constrained p -groups when p is an odd prime.

The organization of the paper is as follows. In Section 2 we exhibit some preliminary facts and tools that will be used in the proofs of the main results of the paper and we introduce the family of normally constrained p -groups. In Section 3 we prove that two generator normally constrained p -groups have a non-inner automorphism of order p . In Section 4 we extend these results to groups generated by more than two elements. Throughout the paper p will be an odd prime, since there are no examples or results in normally constrained 2-groups, except for two generator normally constrained 2-groups that are, as we will see, 2-groups of maximal class.

Throughout the paper, most of the notation is standard and it can be found, for instance, in [16].

2. PRELIMINARIES

Let us start this section recalling some facts about derivations, and some related lemmas, which will be useful to prove the main Theorem 3.5 and Theorem 4.2 of the paper. The reader could be referred to [13] for more details and explicit proofs about derivations.

Definition 2.1. *Let G be a group and let M be a right G -module. A derivation $\delta : G \rightarrow M$ is a function such that*

$$\delta(gh) = \delta(g)^h \delta(h), \text{ for all } g, h \in G.$$

In terms of its properties, it is well-known that a derivation is uniquely determined by its values over a set of generators of G . Let F be a free group generated by a finite subset X and let $G = \langle X : r_1, \dots, r_n \rangle$ be a group whose free presentation is F/R , where R is the normal closure of the set of relations $\{r_1, \dots, r_n\}$ of G . Then a standard argument shows that M is a G -module if and only if M is an F -module on which R acts trivially. Indeed, if we denote by π the canonical homomorphism $\pi : F \rightarrow G$, then the action of F on M is given by $mf = m\pi(f)$, for all $m \in M$ and all $f \in F$. Continuing with the same notation, we have the following results.

Lemma 2.2. *Let M be an F -module. Then every function $f : X \rightarrow M$ extends in a unique way to a derivation $\delta : F \rightarrow M$.*

Lemma 2.3. *Let M be a G -module and let $\delta : G \rightarrow M$ be a derivation. Then $\bar{\delta} : F \rightarrow M$ given by the composition $\bar{\delta}(f) = \delta(\pi(f))$ is a derivation such that $\bar{\delta}(r_i) = 0$ for all $i \in \{1, \dots, n\}$. Conversely, if $\bar{\delta} : F \rightarrow M$ is a derivation such that $\bar{\delta}(r_i) = 0$ for all $i \in \{1, \dots, n\}$, then $\delta(fR) = \bar{\delta}(f)$ defines, uniquely, a derivation on $G = F/R$ to M such that $\bar{\delta} = \delta \circ \pi$.*

In the following lemma, we study the relationship between derivations and automorphisms of a finite p -group.

Lemma 2.4. *Let G be a finite p -group and let M be a normal abelian subgroup of G viewed as a G -module. Then for any derivation $\delta : G \rightarrow M$, we can define uniquely an endomorphism ϕ of G such that $\phi(g) = g\delta(g)$ for all $g \in G$. Furthermore, if $\delta(M) = 1$, then ϕ is an automorphism of G .*

In order to reduce some calculations in terms of commutators, we keep in mind the following result.

Lemma 2.5. *Let F be a free group, p be a prime number and A be an F -module. If $\delta : F \rightarrow A$ is a derivation then,*

- (i) $\delta(F^p) = \delta(F)^p[\delta(F),_{p-1} F]$,
- (ii) if $[A, _i F] = 1$, we have $\delta(\gamma_i(F)) \leq [\delta(F),_{i-1} F]$ for all $i \in \mathbb{N}$.

Proof. Let $x \in F$. We have $\delta(x^p) = \delta(x)^{x^{p-1}+x^{p-2}+\dots+1}$. Since $(x-1)^{p-1} \equiv x^{p-1} + x^{p-2} + \dots + 1 \pmod{p}$, the first assertion follows. Now let us prove the second assertion by induction on i . Clearly, the assertion holds when $i = 1$. Let us suppose, by inductive hypothesis that if $[A, _k F] = 1$ then $\delta(\gamma_k(F)) \leq [\delta(F),_{k-1} F]$ for some $k \in \mathbb{N}$. Let us take any $a \in F$ and any $b \in \gamma_k(F)$, and let us suppose that $[A,_{k+1} F] = 1$. Then

$$\delta([a, b]) = [\delta(a), b][a, \delta(b)][a, b, \delta(a)][a, b, \delta(b)] \in [\delta(F),_k F].$$

□

Now let us introduce the family of normally constrained p -groups, according to [7].

Definition 2.6. *Let G be a finite p -group and let $c(G)$ be its nilpotency class. We say that G is normally constrained (NC for short) if for every $i = 1, \dots, c(G)$ the following equivalent conditions hold true:*

- (i) $\gamma_i(G)$ is unique of its order, for all $i = 1, \dots, c(G)$,
- (ii) if $N \triangleleft G$ then $N \leq \gamma_i(G)$ or $N \geq \gamma_i(G)$, for all $i = 1, \dots, c(G)$,
- (iii) if $x \in G - \gamma_i(G)$ then $\gamma_i(G) \leq \langle x \rangle^G$.

Let us note that factor groups of NC- p -groups are NC, and that the second statement is equivalent to say that if $N \triangleleft G$ then there exists a positive integer i such that $\gamma_i(G) \geq N \geq \gamma_{i+1}(G)$. Now let us list useful properties of these groups whose proofs can be found in [7].

Proposition 2.7. *Let G be an NC- p -group of nilpotency class at least 3. Then $\bar{G} = G/\gamma_3(G)$ is a special p -group (i.e. $Z(\bar{G}) = \Phi(\bar{G}) = \gamma_2(\bar{G})$) of exponent p and $|\gamma_2(G)/\gamma_3(G)|^2 = |G/\gamma_2(G)|$.*

Repeating the argument of the previous result it is possible to prove the following corollary.

Corollary 2.8. *Let G be a NC- p -group of nilpotency class at least 3. Then $\gamma_i(G)/\gamma_{i+2}(G)$ is elementary abelian for all $i \geq 2$.*

Thus every NC- p -group is generated by an even number of elements, and every double quotient of its lower central series has exponent p . Moreover we can obtain a stronger property of such groups, known as *covering property*.

Proposition 2.9. *Let G be a p -group of nilpotency class greater than or equal to 3. The following conditions are equivalent:*

- (i) G is a NC- p -group,
- (ii) for all $i \geq 1$ and for all $x \in \gamma_i(G) - \gamma_{i+1}(G)$ we have $[x, G]\gamma_{i+2}(G) = \gamma_{i+1}(G)$.

Corollary 2.10. *Let G be a NC- p -group of nilpotency class at least 3. Then the upper and lower central series of G coincide.*

The last result implies, clearly, that in a normally constrained p -group G the quotients of the terms of the lower central series are elementary abelian, and so $\gamma_2(G) = \Phi(G)$. Let us finish this section recalling the following result.

Theorem 2.11. *Let G be a NC- p -group of nilpotency class $c(G) \geq 3$ such that $|G : \gamma_2(G)| = p^{2n}$ for some $n \in \mathbb{N}$. Then for all $1 \leq i < c(G)$ we have that $p^n \leq |\gamma_i(G) : \gamma_{i+1}(G)| \leq p^{2n}$.*

3. BERKOVICH CONJECTURE FOR TWO GENERATOR NORMALLY CONSTRAINED p -GROUPS

To develop this section, let us start introducing the family of thin p -groups. Firstly, let us recall that in a group G an antichain is a set of mutually incomparable elements in the lattice of its normal subgroups. It is well-known that, if G is a p -group of maximal class, then the lattice of its normal subgroups consists of $p + 1$ maximal subgroups and of the terms of the lower central series of G . Thus, a p -group of maximal class has only one antichain, which consists of its maximal subgroups. The necessity to extend the family of groups of maximal class to a bigger family of p -groups with a bound on the antichains, leads us to introduce the formal definition of thin p -group. Let us introduce the definition of thin p -groups as in [10].

Definition 3.1. *Let G be a finite p -group. Then G is thin if every antichain in G contains at most $p + 1$ elements.*

The following results about finite thin p -groups are discussed in [8].

Lemma 3.2. *Let G be a finite thin p -group, and let p be an odd prime. If N is a normal subgroup of G , then N is a term of the lower central series of G if and only if N is the unique normal subgroup of its order.*

Remark 3.3. *Let G be a two generator normally constrained p -group. According to Theorem 2.11 all quotients of the terms of the lower central series of G are of order at most p^2 , and by Lemma 3.2 all terms of the lower central series are unique in their order. Thus every antichain in G cannot contain more than $p + 1$ elements, so G is thin. On the other hand, since an elementary abelian p -group is thin if and only if its order is p^2 (see [10]), then every finite thin p -group is a two generator group. Clearly, if G is thin, then*

G is normally constrained, so thin p -groups are exactly normally constrained two generator p -groups.

Forwards, we again mention some already proved results about the existence of non-inner automorphisms of order p in certain specific cases to avoid from now onwards repetitions, not only in the case of finite thin p -groups, but also in the case of finite normally constrained p -groups (which will be studied in the fourth section).

Since Liebeck in [15] proved the existence of at least a non-inner automorphism of order p in all finite p -groups of class 2 for any p odd prime, Abdollahi in [1] proved the existence of such an non-inner automorphism of order 2 in all finite 2-groups of class 2, and Abdollahi, Ghoraishi and Wilkens in [3] did the same in the case of finite p -groups of nilpotency class 3, from now onwards in our study we will deal with finite p -groups of nilpotency class $c = c(G) \geq 4$. On the other hand, since Deaconescu in [12] proved the existence of at least a non-inner automorphism of order p for all finite p -groups which are not strongly Frattinian, we may assume that the finite p -groups G we are interested in, are strongly Frattinian, in other words, that the groups of our interest satisfy $C_G(\Phi(G)) = Z(\Phi(G))$.

Furthermore, as a result due to Abdollahi in [2], we know that if G is a finite p -group such that G has no non-inner automorphisms of order p leaving $\Phi(G)$ elementwise fixed, then $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$. Thus, in view of this previous matter, we may assume that the condition $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$ holds.

Remark 3.4. By Corollary 2.10 and Theorem 2.11 the lower and the upper central series of thin p -groups coincide and all quotients of these series are elementary abelian p -groups of order at most p^2 . Moreover, $Z(G)$ must be cyclic of order p , since we assume that the condition $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$ holds. In particular, the quotients of the lower and upper central of finite thin p -groups have exponent p . Moreover, if G is a finite thin p -group, then $\Phi(G) = \gamma_2(G)$, $Z_2(G) \leq Z(\Phi(G))$, and the property stated in the second part of Proposition 2.9 holds equivalently for terms of the upper central series of G . In addition to this, since in [9] (see Theorem B) it is shown that every finite thin 2-group is of maximal class, in the following we just focus our attention on finite thin p -groups, where p is an odd prime.

Now we are ready to prove the next theorem.

Theorem 3.5. Let G be a finite thin p -group, where p is an odd prime. Then G has a non-inner automorphism of order p .

Proof. Let us denote c the nilpotency class of G with $c \geq 4$. From the assumptions and the consequences of Remarks 3.3 and 3.4 we know that G is a two generator p -group, the lower and the upper central series of G coincide, $\Phi(G) = \gamma_2(G)$, $Z_2(G) \leq Z(\Phi(G))$, $Z(G) \cong C_p$, $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$ and $Z_2(G)/Z(G) \cong C_p \times C_p$. In particular, $[Z_2(G), \gamma_2(G)] = 1$ and $\Omega_1(Z_2(G))$ is an elementary abelian subgroup of G .

Indeed, $G/\gamma_3(G)$ has order p^3 , class 2 and by Proposition 2.7 we may assume that its exponent is p , i.e. $G/\gamma_3(G)$ is an extraspecial p -group of exponent p and order p^3 .

Our goal is to obtain at least an automorphism of G of order p . To do that, firstly, we define an assignment on generators of the free group generated by two elements sending them to $\Omega_1(Z_2(G))$. By Lemma 2.2 it is possible to extend these assignments to a derivation. Secondly, we show that this map preserves the relations defining the quotient $G/\gamma_3(G)$, and then we apply Lemma 2.3 to induce a derivation from $G/\gamma_3(G)$ to $\Omega_1(Z_2(G))$. And finally, we lift this found map to a derivation from G to $\Omega_1(Z_2(G))$ applying Lemma 2.4. In the following paragraphs we describe in detail each of these mentioned steps.

To begin with, let $x \rightarrow u, y \rightarrow v$ be an assignment on generators x, y of the two generator free group F_2 , with $u, v \in \Omega_1(Z_2(G))$. By Lemma 2.3 this assignment extends uniquely to a derivation $\delta : F_2 \rightarrow \Omega_1(Z_2(G))$ such that $\delta(x) = u$ and $\delta(y) = v$. Doing some abuse of notation, we assume that $G/\gamma_3(G)$ corresponds to the presentation $\langle x, y \mid x^p, y^p, [y, x, x], [y, x, y] \rangle$. Next, let us see that the equalities $\delta(x^p) = 1$, $\delta(y^p) = 1$, $\delta([y, x, x]) = 1$ and $\delta([y, x, y]) = 1$ hold. To do it, firstly, let us show that δ is trivial on the p -th powers of elements of F_2 . In fact, considering π' the canonical epimorphism from F_2 to $G/\gamma_3(G)$ we have

$$\begin{aligned} \delta(f^p) &= \delta(f)^{f^{p-1}} \delta(f^{p-1}) = \dots = \delta(f)^{f^{p-1} + \dots + 1} = \delta(f)^{\pi'(f^{p-1} + \dots + 1)} = \\ &= \delta(f)^{\pi'(f)^{p-1} + \dots + 1} = \delta(f)^p [\delta(f), \pi'(f)]^{\binom{p}{2}} = 1, \quad \text{for all } f \in F_2. \end{aligned}$$

Secondly, let us analyze the behaviour of δ on commutators. Since for any $g, h \in G$ it holds that $gh = hg[g, h]$, then applying δ to this previous equality, we get $\delta(gh) = \delta(hg[g, h]) = \delta(hg)^{[g, h]} \delta([g, h])$, and consequently taking into account as well that $[Z_2(G), \gamma_2(G)] = 1$, we get $\delta([g, h]) = \delta(gh)(\delta(hg)^{[g, h]})^{-1} = \delta(gh)(\delta(hg))^{-1} = [\delta(g), h][g, \delta(h)]$, which is an element of $Z(G)$.

Moreover,

$$\begin{aligned} \delta([y, x, y]) &= [\delta([y, x]), y][[y, x], \delta(y)] = 1, \\ \delta([y, x, x]) &= [\delta([y, x]), x][[y, x], \delta(x)] = 1. \end{aligned}$$

Let us underline that the two previous equalities can be obtained as well, applying properly item (ii) of Lemma 2.5.

Now by Lemma 2.3 we can induce a derivation $\bar{\delta}$ from $G/\gamma_3(G)$ to $\Omega_1(Z_2(G))$. The map $\delta' : G \rightarrow \Omega_1(Z_2(G))$ defined for all $g \in G$ by the law $\delta'(g) = \bar{\delta}(g\gamma_3(G))$ is a derivation from G to $\Omega_1(Z_2(G))$. By Lemma 2.4, δ' induces an automorphism ϕ of G by the law $\phi(g) = g\delta'(g)$ for all $g \in G$, leaving, in particular, $\Phi(G)$ elementwise fixed. Clearly, $Z_2(G) \leq Z(\gamma_3(G))$. In fact, this previous inclusion holds since $Z_2(G) \leq Z(\Phi(G))$, $\Phi(G) = \gamma_2(G)$ and $Z_2(G) = \gamma_{c-1}(G) \leq \gamma_3(G)$. Thus this allows us to prove that the automorphism ϕ has order p . In fact, $\phi^p(g) = \phi^{p-1}(g\delta'(g)) = \phi^{p-1}(g)\phi^{p-1}(\delta'(g)) = \phi^{p-1}(g)\delta'(g) = \phi^{p-2}(g\delta'(g))\delta'(g) = \dots = g(\delta'(g))^p = g$, for all $g \in G$.

Repeating this previous construction we produce a set of automorphisms of G of order p , whose size is equal to $|\Omega_1(Z_2(G))|^2$, in other words, whose size is the number of possible choices for the images of the above generators. Next we distinguish the only two possible cases: $Z_2(G) \cong C_p \times C_p \times C_p$ or $Z_2(G) \cong C_{p^2} \times C_p$. In the former case, when $Z_2(G) \cong C_p \times C_p \times C_p$, we can produce p^6 automorphisms of G of order p . However, the number

of inner automorphisms of G induced by elements of $Z_3(G)$ is at most p^4 . Thus, a simple counting argument is enough to say that G has a non-inner automorphism of G of order p , and we get the statement of the Theorem in this case. Otherwise, in the latter case, when $Z_2(G) \cong C_{p^2} \times C_p$, let us choose an assignment $x \rightarrow u, y \rightarrow v$ such that $u, v \in \Omega_1(Z_2(G)) - \{1\}$, u is not central, and let ϕ be the automorphism of G of order p obtained by this assignment. On the other hand, we know that ϕ is inner if and only if there exists an element h_ϕ of $Z_3(G) - Z_2(G)$ such that $h_\phi^p \in Z(G)$, $\phi(g) = g^{h_\phi}$ and $\delta'(g) = [g, h_\phi]$ for all $g \in G$. Under these circumstances, since $Z(G) \leq \Omega_1(Z_2(G))$ we deduce that $[h_\phi, G]Z(G) \leq \Omega_1(Z_2(G)) < Z_2(G)$, which is in contradiction with the analogous covery property of Proposition 2.9. Consequently, this second case does not happen and the statement of the Theorem is proved. \square

4. BERKOVICH'S CONJECTURE FOR NORMALLY CONSTRAINED p -GROUPS

In this section we attack Berkovich's conjecture in normally constrained p -groups, for groups with more than 2 generators. In terms of the following remark, let us see that we can deal with normally constrained p -groups with cyclic center.

Remark 4.1. *As we have established in the previous section, we are dealing with finite p -groups G such that $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$, so we can assume by Theorem 2.11 that the center of G is cyclic of order p (otherwise the quotient $Z_2(G)/Z(G)$ would contradict that above theorem) and that $d(Z_2(G)/Z(G)) = d(G)$. Moreover, from [12] we can assume as well that the second center $Z_2(G)$ is abelian, since otherwise the problem is already solved.*

Theorem 4.2. *Let G be a normally constrained finite p -group generated by more than two elements. Then G has a non-inner automorphism of order p .*

Proof. Let G be a normally constrained p -group d -generated by the elements x_1, \dots, x_d , with $d \geq 3$. Thus according to previous Remark 4.1, the elementary abelian quotient group $Z_2(G)/Z(G)$ is generated by d elements, and $Z_2(G)$ is an elementary abelian subgroup of G generated by $d+1$ elements.

Let us note that the assignment $x\Phi(G) \mapsto [\cdot, x]$ defines an injective homomorphism ϕ from $G/\Phi(G)$ into $\text{Hom}(Z_2(G)/Z(G), Z(G))$, which is actually an isomorphism since $Z_2(G)/Z(G)$ is generated by d elements. Indeed the elements $\phi(x_i)$, where $i = 1, \dots, d$, form a basis of the dual space of $Z_2(G)/Z(G)$. This implies that the intersection of the kernels of $d-1$ of such maps is a vector space one dimensional. Moreover, the order of $K := \bigcap_{i=2}^d C_{Z_2(G)}(x_i)$ is exactly p^2 . Now let us take $u_1 \in K - Z(G)$ and let us define the following assignments:

$$\delta = \begin{cases} x_1 \rightarrow u_1 \\ x_i = 1 \text{ for all } i \geq 2 \end{cases}$$

This δ map is a derivation from the d -generator free group F_d to the F_d -module K , and, arguing as in Theorem 3.5, this derivation induces a new

derivation from

$$G/\Phi(G) = \langle x_1, \dots, x_d \mid x_1^p, \dots, x_d^p, [x_i, x_j] \text{ for all } i, j \rangle$$

to $Z_2(G)$, since for all $i, j \in \{1, \dots, d\}$

$$\delta(x_i^p) = \delta(x_i)^p [\delta(x_i), x_i]^{(p)} = 1$$

and

$$\delta([x_i, x_j]) = [x_i, \delta(x_j)][\delta(x_i), x_j] = 1.$$

Let us note that the previous equalities hold by definition of K . Next, since $K \leq Z(\Phi(G))$, this obtained derivation extends by Lemma 2.4 to an automorphism ϕ , defined by $\phi(g) = g\delta(g)$ for all $g \in G$, whose order is p and that leaves the *Frattini* subgroup elementwise fixed. Moreover, if there exists $h \in Z_3(G) - Z_2(G)$ such that $\phi(g) = g^h = g[g, h]$ for all $g \in G$, then it would deduce that $[h, G] = K$, but this jointly with the fact that $Z(G) < K < Z_2(G)$ contradicts the covering property of G , listed in Proposition 2.9. As a consequence, the obtained automorphism of order p is non-inner, as desired. \square

Remark 4.3. *Let us note that the proof of the above Theorem 4.2 actually works for two generator groups as well. However, it has been a wish of the authors to present to the reader two different ways of producing derivations and solving the problem in the case of two generator normally constrained- p groups, as it has been done in Theorems 3.5 and 4.2, respectively.*

Remark 4.4. *In conclusion, for every odd prime number p , normally constrained p -groups have at least a non-inner automorphism of order p . Moreover, since thin 2-groups are of maximal class, it is well known that they have a non-inner automorphism of order 2. But what happens in the case of normally constrained 2-groups with more than two generators and nilpotency class greater than or equal to 4? The existing bibliography give us neither examples nor results about such groups.*

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